Self attracting diffusions on a sphere and application to a periodic case*

Carl-Erik Gauthier †

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Abstract

This paper proves almost-sure convergence for the self-attracting diffusion on the unit sphere

$$dX(t) = \sigma dW_t(X(t)) - a\int_0^t \nabla_{\mathbb{S}^n} V_{X_s}(X_t)dsdt, \qquad X(0) = x \in \mathbb{S}^n$$

where $\sigma > 0$, $a < 0$, $V_y(x) = \langle x, y\rangle$ is the usual scalar product in $\mathbb{R}^n$, and $(W_t(.))_{t\geqslant 0}$ is a Brownian motion on $\mathbb{S}^n$. From this follows the almost-sure convergence of the real-valued self-attracting diffusion

$$d\vartheta_t = \sigma dW_t + a\int_0^t \sin(\vartheta_t - \vartheta_s)dsdt,$$

where $(W_t)_{t\geqslant 0}$ is a real Brownian motion.

keywords: reinforced process, self-interacting diffusions, asymptotic pseudo-trajectories, rate of convergence.

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1 Introduction

In this paper, we are interested in the asymptotic behaviour of the solutions to the stochastic differential equation (SDE)

$$dX(t) = \sigma dW_t(X(t)) - a\int_0^t \nabla_{\mathbb{S}^n} V_{X_s}(X_t)dsdt, \qquad X(0) = x \in \mathbb{S}^n \tag{1}$$

with $\sigma > 0$, $a \in \mathbb{R}$, $(W_t(.))_t$ is a Brownian motion on the n-dimensional Euclidean unit sphere, $\nabla_{\mathbb{S}^n}$ is the gradient on $\mathbb{S}^n$ and $V_y(x) = \langle x, y\rangle$ where $\langle ., .\rangle$ stands for the canonical scalar product on $\mathbb{R}^{n+1}$.

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†Institut de Mathématiques, Université de Neuchâtel, Rue Émile Argand 11, 2000 Neuchâtel, Switzerland. Email: carl-erik.gauthier@unine.ch

The motivation for investigating (1) lies in the study of the long time behaviour of the real-valued SDE:

$$d\vartheta_t = \sigma dW_t + a\int_0^t \sin(\vartheta_t - \vartheta_s)dsdt, \qquad \vartheta_0 = 0, \tag{2}$$

where $(W_t)_t$ is a real Brownian motion.

Identify ϑ_t to $(\cos(\vartheta_t), \sin(\vartheta_t)) \in \mathbb{S}^1$ and define $d(x,y)$ as the square of the euclidean distance between $(\cos(x), \sin(x))$ and $(\cos(y), \sin(y))$; so that $d(x,y) = 2 - 2\cos(x-y)$. Deriving d with respect to x gives $\partial_x d(x,y) = 2\sin(x-y)$. Therefore, depending on the sign of a, $a\sin(\vartheta_t - y)$ points forward/outward $(\cos(y), \sin(y))$; that is $(\cos(\vartheta_t), \sin(\vartheta_t))$ is attracted by $(\cos(y), \sin(y))$ if $a < 0$ and repelled from $(\cos(y), \sin(y))$ if $a > 0$. Hence, intuitively, $(\cos(\vartheta_t), \sin(\vartheta_t))$ should turn around the circle if $a > 0$ and it should converge to some point if $a < 0$.
Concerning the case $a > 0$, it is proved in the preprint [3] of the author in collaboration with M.Benaïm that

Theorem 1 *The law of*

$$(\cos(\vartheta_t), \sin(\vartheta_t))$$

converges to the uniform law on the circle.

This paper intends to prove that the intuition concerning the attractive case (when $a < 0$) is true. The first main result of this paper is the following result.

Theorem 2 *If $a < 0$, there exists a random variable X_∞ such that $|\vartheta_t - X_\infty| = O(t^{-1/2}\log^{\gamma/2}(t))$, with $\gamma > 1$.*

In 1995, M.Cranston and Y. Le Jan proved a convergence result in [4] in the cases where $a\sin(x)$ is replaced by ax (linear case) or $a \times sgn(x)$, with $a < 0$ (constant case); this last case being extended in all dimension by O.Raimond in [11] in 1997. A few years later, S.Herrmann and B.Roynette weakened the condition of the profile function f around 0 and were still able to get almost sure convergence (see [6]) for the solution to the stochastic differential equation

$$d\vartheta_t = \sigma dW_t + \int_0^t f(\vartheta_t - \vartheta_s)dsdt. \tag{3}$$

Rate of convergence were given in [7] by S.Herrmann and M.Scheutzow. For the linear case, the optimal rate is proven and turns out to be $O(t^{-1/2}\sqrt{t})$.

However, a common fundamental property of these three papers lies in the fact that the associated profile function f is monotone.

We point out that self interacting diffusion involving periodicity has already received some attention in 2002 by M.Benaïm, M.Ledoux and O.Raimond ([1]), *but* in the normalized case; that is, when $\int_0^t \sin(X_t - X_s)ds$ is replaced by $\frac{1}{t}\int_0^t \sin(X_t - X_s)ds$. The interpretation is therefore different. While the drift term of (2) can be "seen" as a summation over $[0,t]$ of the interaction between

the current position X_t and its position at time s and thus an *accumulation of the interacting force* , their drift is then the *average of the interacting force.* The asymptotic behaviour is then given by the following Theorem.

Theorem 3 *(Theorem 1.1, [1], Benaïm, Ledoux, Raimond) Let $(\vartheta_t)_{t\geqslant 0}$ be a solution to the SDE*

$$d\vartheta_t = dW_t + \frac{c}{t}\int_0^t \sin(\vartheta_t - \vartheta_s)dsdt,$$

with initial condition $\vartheta_0 = 0$ and $c \in \mathbb{R}$. Set $X_t = \vartheta_t \bmod 2\pi \in \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ and defined the normalized occupation measure

$$\mu_t = \frac{1}{t}\int_0^t \delta_{X_s} ds.$$

1. *If $c \geqslant -1$, then $\{\mu_t\}$ converges almost surely (for the topology of weak* convergence) toward the normalized Lebesgue measure on $\mathbb{S}^1 \sim [0, 2\pi]$, $\lambda(dx) = \frac{dx}{2\pi}$.*

2. *If $c < -1$, then there exists a constant $\beta(c)$ and a random variable $\varsigma \in [0, 2\pi[$ such that $\{\mu_t\}$ converges almost surely toward the measure*

$$\mu_{c,\varsigma}(dx) = \frac{\exp(\beta(c)\cos(x-\varsigma))}{\int_{\mathbb{S}^1}\exp(\beta(c)\cos(y))\lambda(dy)}\lambda(dx).$$

An intermediate framework between those considered in Theorem 2 and Theorem 3 is to add a time-dependent weight $g(t)$ to the normalized case that increases to infinity when time increases, but "not too fast". In that case, O.Raimond proved the following Theorem

Theorem 4 *(Theorem 3.1, [12], Raimond) Let $(\vartheta_t)_{t\geqslant 0}$ be the solution to the SDE*

$$d\vartheta_t = dW_t - \frac{g(t)}{t}\int_0^t \sin(\vartheta_t - \vartheta_s)dsdt, \tag{4}$$

where g is an increasing function such that $\lim_{t\to} g(t) = \infty$, there exists positives c, t_0 such that for $t \geqslant t_0$, $g(t) \leqslant a\log(t)$ and $|g'(t)| = O(t^{-\gamma})$, with $\gamma \in]0, 1]$. Set $X_t = \vartheta_t \bmod 2\pi$.

Then, there exists a random variable X_∞ in $\mathbb{S}^1$ such that almost surely, $\mu_t = \frac{1}{t}\int_0^t \delta_{X_s} ds$ converges weakly towards δ_{X_∞}.

Interpreting ϑ_t as an angle also provide the link between (1) and (2). Indeed, for $x, y \in \mathbb{S}^n$, one has

$$\|x-y\|^2 = 2 - 2\langle x, y\rangle = 2 - 2\cos(D(x,y)), \tag{5}$$

where $D(.,.)$ is the geodesic distance on $\mathbb{S}^n$. Therefore, Theorem 2 follows from the more general Theorem

Theorem 5 *If $a < 0$, there exists a random variable $X_\infty \in \mathbb{S}^n$ such that*

$$\|X(t) - X_\infty\| = O(t^{-1/2} \log^{\gamma/2}(t)),$$

with $\gamma > 1$ and $\|.\|$ is the standard Euclidean norm in $\mathbb{R}^{n+1}$.

We emphasize that Theorems 1, 3 and 4 are particular cases from more general results proved in the respective papers (Theorem 5 in [3], Theorem 4.5 in [1] and Theorem 3.1 in [12]).

1.1 Reformulation of the problem

From now on, we assume that $a < 0$ and that n is fixed. Since the values of σ and a do not play any particular role, we assume without loss of generality that $\sigma = 1$ and $a = -1$. Thus (1) becomes

$$dX(t) = \sigma dW_t(X(t)) + \int_0^t \nabla_{\mathbb{S}^n} V_{X_s}(X_t) ds dt, \qquad X(0) = x \in \mathbb{S}^n \tag{6}$$

where $V_y(x) = \langle x, y\rangle =: V(y, x)$. Since V satisfies Hypothesis 1.3 and 1.4 in [1], then (6) admits a unique strong solution by Proposition 2.5 in [1]. We emphasize that equation (2) admits a unique strong solution because the function $\sin(.)$ is Lipschitz continuous (see for example Proposition 1 in [6]). We shall begin by clarifying the two quantities that appear in (6). Firstly, by Example 3.3.2 in [8], we have

$$dW_t(X(t)) = dB(t) - X_t\langle X_t, \circ dB_t\rangle, \tag{7}$$

where $\circ$ stands for the Stratonovitch integral and $(B(t))_{t\geqslant 0}$ is a $(n+1)$ dimensional Brownian motion.

Secondly, for a function $F : \mathbb{R}^{n+1} \to \mathbb{R}$, we have

$$\nabla_{\mathbb{S}^n}(F_{|_{\mathbb{S}^n}})(x) = \nabla_{\mathbb{R}^{n+1}} F(x) - \langle x, \nabla_{\mathbb{R}^{n+1}} F(x)\rangle x;\ x \in \mathbb{S}^n. \tag{8}$$

Hence equation (6) rewrites

$$dX(t) = dB(t) - X(t)\langle X(t), \circ dB(t)\rangle + \int_0^t (X_s - \langle X_t, X_s\rangle X_t) ds dt, \tag{9}$$

Following the same idea as in [3], we set $U_t := \int_0^t X(s) ds \in \mathbb{R}^{n+1}$ in order to get the SDE on $\mathbb{S}^n \times \mathbb{R}^{n+1}$:

$$\begin{cases} dX(t) = dB(t) - X(t)\langle X(t), \circ dB(t)\rangle + \sum_{j=1}^{n+1} U_j(t)[e_j - X_j(t)X(t)]dt \\ dU(t) = X(t)dt \end{cases} \tag{10}$$

with initial condition $(X(0), U(0)) = (x, 0)$ and $e_1, \cdots, e_{n+1}$ stand for the canonical basis of $\mathbb{R}^{n+1}$.

A first property is

Lemma 6 *$d(\|U(t)\|^2) = 2\langle U(t), X(t)\rangle dt$, where $\|.\|$ stands for the standard Euclidean norm in $\mathbb{R}^{n+1}$.*

Proof. It is a direct consequence of Itô's Formula. ■

The paper is organised as follow. In Section 2, we present the detailed strategy used for proving Theorem 5 whereas the more technical proofs are presented in Section 3.

2 Guideline of the proof of Theorem 2

Set $R_t = \|U(t)\|$ and define $V_t \in \mathbb{S}^n$ and $\Theta_t \in [-1, 1]$ as follows:

$$V(t) = \begin{cases} U(t)/R_t \text{ if } R_t > 0 \\ X(t) \text{ otherwise} \end{cases} \tag{11}$$

and

$$\Theta_t = \langle V_t, X(t)\rangle \tag{12}$$

Since $(\int_0^t \langle V(s) - \Theta_s X(s), dB(s))_{t\geqslant 0}$ is a real valued martingale, then by the Martingale representation Theorem by a Brownian motion (see Theorem 5.3 in [5]), there exists a real valued Brownian motion $(W_t)_t$ such that

$$\int_0^t \langle V(s) - \Theta_s X(s), dB(s)\rangle = \int_0^t \sqrt{1 - \Theta_s^2} dW_s. \tag{13}$$

Lemma 7 *$((\Theta_t, R_t))_{t\geqslant}$ is solution to*

$$\begin{cases} dY_t = \sqrt{1 - Y_t^2} dW_t + [(r_t + \frac{1}{r_t})(1 - Y_t^2) - \frac{n}{2} Y_t]dt \\ dr_t = Y_t dt \end{cases} \tag{14}$$

whenever $R_t > 0$.

Proof. From Lemma 6, we deduce

$$dR_t = \Theta_t dt. \tag{15}$$

Applying Itô's Formulae to $\langle U(t), X(t)\rangle$ gives

$$\begin{aligned} \langle U(t), X(t)\rangle &= t + \int_0^t (\|U(s)\|^2 - \langle U(s), X(s)\rangle^2) ds + \int_0^t \langle U(s) - \langle X(s), U(s)\rangle X(s), \circ dB_s\rangle. \\ &= \int_0^t (1 - \frac{n}{2}\langle X(s), U(s)\rangle ds + \int_0^t (\|U(s)\|^2 - \langle U(s), X(s)\rangle^2) ds \\ &\quad + \int_0^t \langle U(s) - \langle X(s), U(s)\rangle X(s), dB_s\rangle \end{aligned} \tag{16}$$

Since $\langle U(t), X(t)\rangle = R_t\Theta_t$, we have by Itô's Formulae

$$d\Theta_t = \frac{1}{R_t}(d(\langle U(t), X(t)\rangle - \Theta_t dR_t). \tag{17}$$

Combining (13), (15) and (16), we obtain

$$\begin{aligned} d\Theta_t &= (\frac{1}{R_t} - \frac{n}{2}\Theta_t + R_t(1-\Theta_t^2))dt - \frac{\Theta_t^2}{R_t}dt + \langle V(t) - \Theta_t X(t), dB(t)\rangle \\ &= [(R_t + \frac{1}{R_t})(1-\Theta_t^2) - \frac{n}{2}\Theta_t]dt + \sqrt{1-\Theta_t^2}dW_t \end{aligned} \tag{18}$$

■ A first important result, whose proof is postponed to Section 3, is

Lemma 8 *We have* $\liminf_{t\to\infty} \frac{R_t}{\sqrt{t}} \geqslant 1$ *almost-surely.*

From this Lemma, we prove in Section 3

Lemma 9 *We have that* $(\Theta_t)_{t\geqslant 0}$ *and* $(\frac{R_t}{t})_{t>0}$ *converge almost surely to* 1*. Furthermore the rate of convergence is* $O(t^{-1}\log^\gamma(t))$*, with* $\gamma > 1$*.*

From (15) and the definition of $U(t)$, it follows from Itô's Formulae

$$dV_t = \frac{1}{R_t}(X(t) - \Theta_t V_t)dt. \tag{19}$$

Lemma 10 V_t *converges almost surely.*

Proof. Since

$$\|X(t) - \Theta_t V_t\| = \sqrt{1-\Theta_t^2}, \tag{20}$$

it follows from (19) that

$$\frac{1}{R_t}\|X(t) - \Theta_t V_t\| = O(t^{-3/2}\log^{\gamma/2}(t)), \tag{21}$$

which is an integrable quantity. ■

We can now prove the main result.

Proof of Theorem 2.

By Lemmas 9 and 10, there exists a random variable $X_\infty \in \mathbb{S}^n$ such that $\lim_{t\to\infty}\Theta_t V_t = X_\infty$. The rate of convergence follows from the triangle inequality, (20), (21) and Lemma 9.

3 Proofs of Lemma 8 and Lemma 9

3.1 Proof of Lemma 8.

Set $M_t = -2\int_0^t \langle U(s) - \langle X(s), U(s)\rangle X(s), dB_s\rangle$. Then its quadratic variation is

$$\begin{aligned} \langle M\rangle_t &= 4\int_0^t \|U(s) - \langle X(s), U(s)\rangle X(s)\|^2 ds. \\ &= 4\int_0^t (\|U(s)\|^2 - \langle U(s), X(s)\rangle^2)ds. \end{aligned} \tag{22}$$

So, from (16) and Lemma 6, we obtain

$$\exp(-2(\langle X(t),U(t)\rangle+\frac{1}{4}R_t^2))=\exp(-2t)\exp(M_t-\frac{1}{2}\langle M_t\rangle), \tag{23}$$

Since

$$\langle M\rangle_t \leqslant 4\int_0^t \|U(s)\|^2 ds \leqslant 8\int_0^t \|U(0)\|^2+s^2 ds = 8(\frac{t^3}{3}+\|U(0)\|^2 t), \tag{24}$$

M_t satisfies the Novikov Condition (see [10], Chapter V, section D, page 198). Therefore

$$\mathcal{E}_M(t) := \exp(M_t - \frac{1}{2}\langle M\rangle_t)$$

is a martingale having 1 as expectation.

Thus

$$\mathbb{E}(\exp(-2(\langle X(t),U(t)\rangle+\frac{1}{4}R_t^2))=\exp(-2t). \tag{25}$$

Consequently,

$$\exp(-2(\langle X(t),U(t)\rangle+\frac{1}{4}R_t^2))=\exp(-2(R_t\Theta_t+\frac{1}{4}R_t^2))$$

converges almost surely to 0 by the Markov inequality and the Borel-Cantelli Lemma. Furthermore, for all $0<c<2$, there exists a random variable T such that almost surely

$$\exp(-2(R_t\Theta_t+\frac{1}{4}R_t^2))\leqslant \exp(-ct), \qquad \forall t\geqslant T. \tag{26}$$

Hence, for $t\geqslant T$,

$$R_t\Theta_t+\frac{1}{4}R_t^2\geqslant \frac{c}{2}t.$$

Therefore, by choosing $c=1$, we obtain

$$R_t\geqslant -1+\sqrt{t} \text{ for } t\geqslant T. \tag{27}$$

This concludes the proof.

3.2 Proof of Lemma 9.

Before starting the proof of Lemma 9, let us recall the Definition of an *asymptotic pseudotrajectory* introduced by Benaïm and Hirsch in [2].

Definition 11 *Let (M,d) be a metric space and Φ a semiflow; that is*

$$\Phi:\mathbb{R}_+\times M\to M:(t,x)\mapsto \Phi(t,x)=\Phi_t(x)$$

is a continuous map such that

$$\Phi_0 = Id \textit{ and } \Phi_{t+s}=\Phi_t\circ\Phi_s$$

for all $s, t \in \mathbb{R}_+$.
A continuous function $X : \mathbb{R}_+ \to M$ is an asymptotic pseudotrajectory for Φ if

$$\lim_{t\to\infty} \sup_{0\leqslant h\leqslant T} d(X(t+h), \Phi_h(X(t))) = 0 \tag{28}$$

for any $T > 0$. In words, it means that for each fixed $T > 0$, the curve $X : [0, T] \to M : h \mapsto X(t+h)$ shadows the Φ-trajectory over the interval $[0, T]$ with arbitrary accuracy for sufficiently large t.
If X is a continuous random process, then X is an almost-surely asymptotic pseudotrajectory for Φ if (28) holds almost-surely.

Theorem 12 *(Theorem 1.2 in [2]) Suppose that $X([0,\infty))$ has compact closure in M and set $L(X) = \bigcap_{t\geqslant 0} \overline{X([t,\infty))}$. Let A be an attractor for Φ with basin W. If $X_{t_k} \in W$ for some sequence $t_k \to \infty$, then $L(X) \subset A$.*

The following result gives a sufficient condition between a SDE on $\mathbb{R}$ and the related ODE when the diffusion term vanishes.

Theorem 13 *(Proposition 4.6 in [2]) Let $g : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function and $\sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ a continuous function. Assume there exists a non-increasing function $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\sigma^2(t,x) \leqslant \varepsilon(t)$ for all (t,x) and such that*

$$\forall k > 0, \int_0^\infty \exp(-k/\varepsilon(t))dt < \infty^1. \tag{29}$$

Then, all solution of

$$dx_t = g(x_t)dt + \sigma(t, x_t)dB_t$$

is with probability 1 an asymptotic pseudotrajectory for the flow induced by the ODE $\dot{X}(t) = g(X(t))$. Furthermore, for all $T > 0$, there exist constant $C, C(T) > 0$, such that for all $\beta > 0$,

$$\mathbb{P}(\sup_{0\leqslant h\leqslant T} |x_{t+h} - \Phi_h(x_t)| \geqslant \beta) \leqslant C \exp(-(\beta C(T))^2/\varepsilon(t)). \tag{30}$$

Remark 14 *The same result holds if $(x_t)_{t\geqslant 0}$ is solution to the SDE*

$$dx_t = g(x_t)dt + \sigma(t, x_t)dB_t + \delta(t)h(x_t)dt,$$

where h is a bounded function and δ is a non-negative function with $\lim_{t\to\infty} \delta(t) = 0$. In that case, for all $T > 0$, there exist constant $C, C(T,h) > 0$, such that for all $\beta > 0$,

$$\mathbb{P}(\sup_{0\leqslant h\leqslant T} |x_{t+h} - \Phi_h(x_t)| \geqslant \beta) \leqslant C \exp(-(\beta - \sup_{s\in[t,t+T]} \delta(s))^2 C(T,h)/\varepsilon(t)). \tag{31}$$

[1] For example $\varepsilon(t) = O(1/(\log(t))^\alpha)$ with $\alpha > 1$.

Proof of Lemma 9:

The proof is divided into two parts.

Proof of the convergence:

First we prove that Θ_t converges almost surely to 1. Recall that

$$d\Theta_t = \sqrt{1-\Theta_t^2}dW_t + [(R_t + \frac{1}{R_t})(1-\Theta_t^2) - \frac{n}{2}\Theta_t]dt. \tag{32}$$

Define $\alpha(t) = (\frac{3}{2}t)^{\frac{2}{3}}$ so that $\dot{\alpha}(t) = \alpha^{-\frac{1}{2}}(t)$. Set $Z_t := \Theta_{\alpha(t)}$ and $M_t = W_{\alpha(t)}$. Thus $(M_t)_t$ is a martingale with respect to the filtration $\mathcal{G}_t = \sigma\{W_s \mid 0 \leqslant s \leqslant \alpha(t)\}$, whose quadratic variation at time t is $\alpha(t) = \int_0^t (\sqrt{\dot{\alpha}(s)})^2 ds$.
Then by the Theorem of Martingale Representation by a Brownian motion (see Theorem 5.3 in [5]), there exists a Brownian motion $(B_t^{(\alpha)})_t$ adapted to $(\mathcal{G}_t)_t$ such that

$$M_t = \int_0^t \sqrt{\dot{\alpha}(s)} dB_s^{(\alpha)}.$$

Then

$$\begin{aligned} Z_t &= \int_0^{\alpha(t)} \sqrt{1-\Theta_s^2}dW_s + \int_0^{\alpha(t)} (R_s + \frac{1}{R_s})(1-\Theta_s^2)ds - \frac{n}{2}\int_0^{\alpha(t)} \Theta_s ds \qquad (33) \\ &= \int_0^t \sqrt{\dot{\alpha}(s)}\sqrt{1-Z_s^2}dB_s^{(\alpha)} - \int_0^t \frac{R_{\alpha(s)} + \frac{1}{R_{\alpha(s)}}}{\sqrt{\alpha(s)}}ds - \frac{n}{2}\int_0^t \dot{\alpha}(s) Z_s ds, \end{aligned}$$

For $y \in [-1,1]$, let $(Y_t^y)_{t\geqslant 0}$ be the solution to the SDE taking values in $[-1,1]$

$$\begin{cases} dY_t^y = \sqrt{\dot{\alpha}(t)}\sqrt{1-(Y_t^y)^2}dB_t^{(\alpha)} + [\frac{1}{2}(1-(Y_t^y)^2) - \frac{n}{2}\dot{\alpha}(t)Y_t^y]dt \\ Y_0^y = y \end{cases} \tag{34}$$

We divide the proof of the convergence in two steps. In the first one, we prove for all $y \in [-1,1]$, Y_t^y converges almost surely to 1; and then prove the convergence of Z_t to 1 in the second one.

Step I: Let $y \in [-1,1]$. In order to lighten the notation, we omit the superscript y in Y_t^y in this step. We start by proving that Y_t is an asymptotic pseudotrajectory for the flow induced by the ODE

$$\dot{x} = \frac{1}{2}(1-x^2). \tag{35}$$

In order to achieve it, we use Theorem 13. Since $x \mapsto (1-x^2)$ is Lipschitz continuous on $[-1,1]$ and that $Z_t \in [-1,1]$ for all $t \geqslant 0$, it remains to prove the hypothesis concerning the noise term.

Set

$$\varepsilon(t) := \dot{\alpha}(t) = (\frac{3}{2}t)^{-\frac{1}{3}}. \tag{36}$$

We have to show that $\varepsilon(t)$ satisfies (29). But this is immediate because for all $k > 0$

$$\int_0^\infty \exp(-kt^{1/3})dt < \infty. \tag{37}$$

Since $Y_t \in [-1,1]$ for all $t \geqslant 0$, it is clear that the condition in Remark 14 is satisfied. Consequently, by Theorem 13, $(Y_t)_t$ is an asymptotic pseudotrajectory of (35).

Because $\{1\}$ is an attractor for the flow induced by (35) with basin $]-1,1]$ and that almost-surely $Y_t \in]-1,1]$ infinitely often, then

$$\lim_{t\to\infty} Y_t = 1 \text{ a.s.} \tag{38}$$

Step II: Our goal is to prove

$$\mathbb{P}(\lim_{t\to\infty} Z_t = 1) = 1. \tag{39}$$

Define the stopping times $\tau_0 = 0$,

$$\tau_j = \inf(t > \sigma_j \mid \frac{R_{\alpha(t)}}{\sqrt{\alpha(t)}} = \frac{1}{2}), \quad j \geqslant 1 \tag{40}$$

and

$$\sigma_j = \inf(t > \tau_{j-1} \mid \frac{R_{\alpha(t)}}{\sqrt{\alpha(t)}} = \frac{3}{4}), \quad j \geqslant 1. \tag{41}$$

By Lemma 8, we have

$$\mathbb{P}(\bigcup_{j\geqslant 1}\{\tau_j = \infty\}) = 1 \text{ and } \mathbb{P}(\sigma_j < \infty \mid \tau_{j-1} < \infty) = 1. \tag{42}$$

Let start by computing $\mathbb{P}(\lim_{t\to\infty} Z_t = 1,\ \tau_j = \infty)$. For $s \in [\sigma_j, \tau_j]$, we have

$$\frac{R_{\alpha(s)} + \frac{1}{R_{\alpha(s)}}}{\sqrt{\alpha(s)}} \geqslant \frac{1}{2}.$$

So, by a comparison result (see Theorem 1.1, Chapter VI in [9]),

$$\mathbb{P}(Z_{(t+\sigma_j)\wedge\tau_j} \geqslant Y^{Z_{\sigma_j}}_{(t+\sigma_j)\wedge\tau_j},\ \forall t \geqslant 0) = 1. \tag{43}$$

As a consequence, we have

$$\begin{aligned} \mathbb{P}(\lim_{t\to\infty} Z_t = 1,\ \tau_j = \infty,\ \sigma_j < \infty) &\geqslant \mathbb{P}(\lim_{t\to\infty} Y^{Z_{\sigma_j}}_{t+\sigma_j} = 1,\ \tau_j = \infty,\ \sigma_j < \infty) \\ &= \mathbb{P}(\tau_j = \infty,\ \sigma_j < \infty). \end{aligned} \tag{44}$$

where the last equality follows from Step I. Because $\{\sigma_j = \infty\} \subset \{\tau_j = \infty\}$, it follows from (42)

$$\mathbb{P}(\lim_{t\to\infty} Z_t = 1,\ \tau_j = \infty) = \mathbb{P}(\lim_{t\to\infty} Z_t = 1,\ \tau_{j-1} = \infty) + \mathbb{P}(\tau_j = \infty,\ \sigma_j < \infty). \tag{45}$$

Thus,

$$\begin{aligned}\mathbb{P}(\lim_{t\to\infty} Z_t = 1,\ \tau_j = \infty) &= \sum_{k=1}^{j} \mathbb{P}(\tau_k = \infty,\ \sigma_k < \infty) \\ &= \mathbb{P}(\tau_j = \infty). \end{aligned} \tag{46}$$

Since $(\{\tau_j = \infty\})_{j\geqslant 0}$ is an increasing family of event, we obtain from (42)

$$\begin{aligned}\mathbb{P}(\lim_{t\to\infty} Z_t = 1) &= \lim_{j\to\infty} \mathbb{P}(\lim_{t\to\infty} Z_t = 1,\ \tau_j = \infty) \\ &= \lim_{j\to\infty} \mathbb{P}(\tau_j = \infty) \\ &= 1. \end{aligned} \tag{47}$$

Consequently, Θ_t converges a.s to 1. Therefore,

$$\frac{R_t}{t} = \frac{1}{t}(R_0 + \int_0^t \Theta_s ds) \tag{48}$$

converges a.s to 1.

Proof of the rate of convergence:

In view of the previous part, it suffices to determine the rate of convergence to 1 of Y_t^y, where $\alpha(t)$ is now $\alpha(t) = \sqrt{2t}$.

For $c > 0$, let $(Y_c^y(t))_{t\geqslant 0}$ be the solution to the SDE

$$(SDE(c,y)) \begin{cases} dY_c^y(t) = \sqrt{c}\sqrt{\dot\alpha(t)}\sqrt{1-(Y_c^y(t))^2}dB_t^{(\alpha)} + [\frac{1}{2}(1-(Y_c^y(t))^2) - \frac{n}{2}\dot\alpha(t)Y_c^y(t)]dt \\ Y_c^y(t) = y \end{cases} \tag{49}$$

with $\alpha(t) = \sqrt{2t}$. Note that Y_t^y is a strong solution to $SDE(1,y)$. Let $(\varphi(t))_{t\geqslant 0}$ be the solution to the SDE

$$\begin{cases} d\varphi(t) = \sqrt{n\dot\alpha(t)}d\bar{B}_t - \frac{1}{2}\sin(\varphi(t))dt \\ \varphi(0) = \varphi_0 \end{cases} \tag{50}$$

where $(\bar{B}_t)_t$ is a real Brownian motion. Then $(\cos(\varphi(t)))_{t\geqslant 0}$ is a weak solution to $SDE(n, \cos(\varphi_0))$. So, if $|\varphi(t)| = O(\Delta(t))$, then by a Taylor expansion, $1 - \cos(\varphi(t)) = O(\Delta^2(t))$.

Since for all $c > 0$, $Y_1^y(t)$ and $Y_c^y(t)$ converge to 1 with a similar rate (by using the fact that $c\dot\alpha(t) = \dot\alpha(\frac{t}{c^2})$), it suffices to find out the rate of $Y_n^y(t)$.
The goal now consists on identifying such a function $\Delta(.)$. We proceed in three steps. First, we assume that there exists a positive decreasing function δ that converges to 0 such that

i. for all $\nu > 0$ and $0 < \kappa < 1$, $\lim_{t\to\infty} \delta(t)\exp(\nu t) = \infty$ and there exists $0 < c(\kappa) < \infty$ such that $\lim_{t\to\infty} \frac{\delta(\kappa t)}{\delta(t)} = c(\kappa)$,

ii. for all $T > 0$, $\sup_{0 \leqslant h \leqslant T} |\varphi(t+h) - \Phi_h(\varphi(t))| = O(\delta(t))$, where Φ is the flow induced by the ODE $\dot{x} = -\frac{1}{2}\sin(x)$

and prove that $|Z_t| = O(\delta(t))$. Next, we prove the existence of such a function δ and conclude the proof in the last step.

Step I: Let $T > 0$. From the convergence of $\cos(\varphi(t,\omega))$ to 1, there exists $t_1(\omega)$ such that, without loss of generality, $\varphi(t,\omega) > 0$ for all $t > t_1(\omega)$. Since $|\sin(x)| \geqslant \frac{2}{\pi}x =: \lambda x$ for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, then for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$,

$$0 \leqslant |\Phi_h(x)| \leqslant e^{-\lambda h}|x|.$$

Thus,

$$\begin{aligned} |\varphi(t+T)| &\leqslant |\varphi(t+T) - \Phi_T(\varphi(t))| + |\Phi_T(\varphi(t))| \\ &\leqslant \delta(t) + e^{-\lambda T}|\varphi(t)|. \end{aligned} \tag{51}$$

Set $y_k := |\varphi(kT)|$, $\delta_k = \delta(kT)$ and $\gamma := e^{-\lambda T} < 1$. Thanks to (51), we have

$$y_{k+1} \leqslant \delta_k + \gamma y_k. \tag{52}$$

Hence

$$y_{k+r} \leqslant \gamma^k y_r + \sum_{j=0}^{k-1} \delta_{k+r-j}\gamma^j. \tag{53}$$

By assumption i., $\delta_k \gamma^{-k}$ converges to ∞. Thus

$$y_k = O(\delta_k). \tag{54}$$

For $kT \leqslant t \leqslant (k+1)T$, we have

$$\begin{aligned} |\varphi(t)| &\leqslant |\Phi_{t-kT}(\varphi(kT)) - \varphi(t)| + |\Phi_{t-kT}(\varphi(kT))| \\ &\leqslant \delta_k + e^{-\lambda(t-kT)}y_k \\ &= O(\delta_k). \end{aligned} \tag{55}$$

Hence,

$$|\varphi(t)| = O(\delta(t)).$$

Step II: Let $T > h > 0$. From Theorem 13, there exists constants $C, C(T)$ such that

$$\mathbb{P}(\sup_{0 \leqslant h \leqslant T} |\varphi(t+h) - \Phi_h(\varphi(t))| > \beta) \leqslant C\exp(-\frac{(\beta C(T))^2}{n\dot{\alpha}(t)}). \tag{56}$$

Defining $\beta_{\gamma,T}(t) := \frac{1}{C(T)}(\dot{\alpha}^{1/2}(t)\log^{\frac{\gamma}{2}}(1+t))$, with $\gamma > 1$, we get

$$\mathbb{P}(\sup_{0 \leqslant h \leqslant T} |\varphi(t+h) - \Phi_h(\varphi(t))| > \beta_{\gamma,T}(t)) \leqslant C\exp(-\log^{\gamma}(1+t)) \tag{57}$$

Since

$$\int_0^\infty \exp(-\log^\gamma(1+t))dt < \infty \tag{58}$$

we deduce by the Borel-Cantelli Lemma that almost-surely

$$\begin{aligned}\sup_{0\leqslant h\leqslant T}|\varphi(t+h)-\Phi_h(\varphi(t))| &= O(\beta_{\gamma,T}(t)) \\ &= O(\dot{\alpha}^{1/2}(t)\log^{\frac{\gamma}{2}}(t)) \\ &= O(t^{-1/4}\log^{\frac{\gamma}{2}}(t)). \end{aligned}\tag{59}$$

Step III: Since $\alpha(t)=\sqrt{2t}$, it is clear that assumptions i. and ii. are satisfied. So, from Steps I and II,

$$|\varphi_t| = O(t^{-1/4}\log^{\frac{\gamma}{2}}(t)). \tag{60}$$

Therefore,

$$|Y_t| = O(t^{-1/2}\log^{\gamma}(t)). \tag{61}$$

Consequently,

$$|Z_t| = O(t^{-1/2}\log^{\gamma}(t)). \tag{62}$$

and so

$$|\Theta_t| = O(t^{-1}\log^{\gamma}(t)). \tag{63}$$

4 Conclusion

The motivating model of this work was the real-valued self-attracting diffusion

$$dX_t = \sigma dW_t + a\int_0^t \sin(X_t - X_s)dsdt, \qquad X_0 = 0.$$

Seeing it as an angle, it turned out that the almost sure convergence of X_t was an immediate consequence of the more general diffusion on the $n-$dimensional unit sphere $\mathbb{S}^n$

$$dX(t) = \sigma dW_t(X(t)) - a\int_0^t \nabla_{\mathbb{S}^n} V_{X_s}(X_t)dsdt, \qquad X(0) = x \in \mathbb{S}^n$$

with $V_y(x) = \langle x, y\rangle$.

It would now be interesting to study the self-reinforced diffusion

$$dX_t = \sigma dW_t + \sum_{k=1}^n k a_k \int_0^t \sin(k(X_t - X_s))dsdt,$$

where the coefficient $a_k \neq 0$ are such that $\sum_{k=1}^n k^2 a_k < 0$.

Because $\sum_{k=1}^n k^2 a_k = (\sum_{k=1}^n k a_k \sin(k.))'(0)$ and that it has to play a more and more important role if $(X_t)_t$ localizes, it sounds reasonable to formulate the following conjecture:

Conjecture 15 *Let $(X_t)_{t\geqslant 0}$ be the solution to the SDE*

$$dX_t = \sigma dW_t + \sum_{k=1}^{n} k a_k \int_0^t \sin(k(X_t - X_s))dsdt, \ X_0 = x.$$

If $\sum_{k=1}^n k^2 a_k < 0$ (resp. $\sum_{k=1}^n k^2 a_k > 0$), then X_t converges almost-surely (resp. $\limsup_t X_t > \liminf_t X_t$).

A Almost sure convergence for the linear case

Since $\sin(x) \sim x$ on a small neighbourhood of 0, the aim to this appendix is to show that the rate of convergence that we obtain is quasi-optimal.

Proposition 16 *Let X_t be the solution of the SDE*

$$dX_t = -\lambda X_t dt + \sqrt{\varepsilon(t)} dB_t, \tag{64}$$

with initial condition $X_0 = x$. $(B_t)_t$ stands for a real Brownian motion and $\lambda > 0$. Assume that $\varepsilon(.)$ is a non-increasing positive continuous function such that $\lim_{t\to\infty} \varepsilon(t) = 0$.

Set $\sigma_t^2 = e^{-2\lambda t} \int_0^t e^{2\lambda s} \varepsilon(s) ds$.

1. *If $\int_0^t e^{2\lambda s}\varepsilon(s)ds < \infty$, then $|X_t| = O(e^{-\lambda t})$*

2. *If $\int_0^t e^{2\lambda s}\varepsilon(s)ds = \infty$, then $|X_t| = O(\sigma_t \sqrt{\log(t)})$. In particular,*

   (a) *If $\varepsilon(t) = O(e^{-2\alpha t})$ with $\alpha < \lambda$, then $\sigma_t^2 = O(e^{-2\alpha t})$*

   (b) *If $\varepsilon(t) = O(t^{-\alpha})$, $\alpha > 0$, then $\sigma_t^2 = O(\varepsilon(t))$*

Proof. We assume without loss of generality that $\varepsilon(0) < \infty$. The solution of Equation (64) is

$$X_t = \exp(-\lambda t)(x + \int_0^t \exp(\lambda s)\sqrt{\varepsilon(s)} dB_s =: \exp(-\lambda t)(x + M_t). \tag{65}$$

The quadratic variation of M_t is then

$$\langle M \rangle_t = \int_0^t \exp(2\lambda s)\varepsilon(s)ds.$$

For the first assertion of the proposition, we then have that $t \mapsto M_t$ is bounded. The conclusion follows from (65).

Concerning the second statement, we have, by the Dubins-Schwarz Theorem (see Theorem 4.6 in [10]) with the law of Iterated Logarithm for Brownian motion (see Theorem 9.23, Chapter 2 in [10]),

$$\limsup_{t\to\infty} \frac{|M_t|}{\sqrt{2\langle M\rangle_t \log\log\langle M\rangle_t}} = 1 \tag{66}$$

almost surely. Thus $\frac{|M_t|}{\sqrt{2\langle M\rangle_t \log\log\langle M\rangle_t}}$ is bounded.
Since $\langle M\rangle_t \leqslant \frac{\varepsilon(0)}{2\lambda}(\exp(2\lambda t)-1)$, we have

$$\log\log\langle M\rangle_t = O(\log(t)). \tag{67}$$

Thus,

$$\begin{aligned}\sqrt{2\langle M\rangle_t \log\log\langle M\rangle_t}\exp(-\lambda t) &= \sqrt{\exp(-2\lambda t)\langle M\rangle_t}\sqrt{2\log\log\langle M\rangle_t} \\ &= O(\sigma_t\sqrt{\log(t)}). \end{aligned}\tag{68}$$

Part (2.(a)) follows from the definition of σ_t^2. For part (2.(b)), we have

$$\begin{aligned}\exp(-2\lambda t)\int_0^t \exp(2\lambda s)\varepsilon(s)ds &= \exp(-2\lambda t)\int_0^{t-\sqrt{t}} \exp(2\lambda s)\varepsilon(s)ds \\ &\quad + \exp(-2\lambda t)\int_{t-\sqrt{t}}^t \exp(2\lambda s)\varepsilon(s)ds \\ &\leqslant \frac{\varepsilon(0)}{2\lambda}\exp(-2\lambda t)(\exp(2\lambda t - 2\lambda\sqrt{t})-1) \\ &\quad +\frac{1}{2\lambda}\exp(-2\lambda t)\varepsilon(t-\sqrt{t})\exp(2\lambda t) \\ &\quad -\frac{1}{2\lambda}\exp(-2\lambda t)\varepsilon(t-\sqrt{t})\exp(2\lambda t - 2\lambda\sqrt{t}).\end{aligned}$$

Thus

$$\begin{aligned}\exp(-2\lambda t)\langle M\rangle_t &\leqslant \frac{\varepsilon(0)}{2\lambda}(\exp(-2\lambda\sqrt{t}) - \exp(-2\lambda t)) \\ &\quad +\frac{1}{2\lambda}\varepsilon(t-\sqrt{t})(1-\exp(-2\lambda\sqrt{t})).\end{aligned}$$

Because for all $\beta \geqslant 0$, $(t-\sqrt{t})^\beta$ is equivalent to t^β when $t\to\infty$, we obtain the existence of a constant C such that

$$\exp(-2\lambda t)\langle M\rangle_t \leqslant C^2\varepsilon(t) \tag{69}$$

for t large enough.
■

Remark 17 *To ensure almost sure convergence to* 0*, the maximal noise intensity has to be* $\varepsilon(t) = O(1/\log(t)^\alpha)$*, with* $\alpha > 1$*.*

A direct consequence of Proposition 16.2.*ii* is

Corollary 18 *If* $\varepsilon(t) = (1+t)^{-\alpha}$ *with* $\alpha > 0$*, then the solution* $(X_t)_{t\geqslant 0}$ *of (64) satisfies* $|X_t| = O(t^{-\frac{\alpha}{2}}\sqrt{\log(t)})$*.*

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References

[1] Benaïm, M., Ledoux, M., Raimond, O. : *Self-interacting diffusions* , Probab. Theory Relat. Fields. Vol 122, Issue 1: 1-41. (2002).

[2] Benaïm, M., Hirsch, M.W.: *Asymptotic Pseudo-Trajectories and Chain-recurrent flows, with Applications* , Journal of Dynamics and Differential Equations. Vol 8, No 1 pp 141-174. (1996).

[3] Benaïm, M., Gauthier, C.E.: *Self repelling diffusions on a Riemannian manifolds*, preprint available on arXiv1505.05664 (2015)

[4] Cranston, M., Le Jan, Y.: *Self attracting diffusions : Two case studies*, Math. Ann. 303(1): 87-93 (1995)

[5] Doob, J.L.: *Stochastic Processes*, John Wiley, New-York, 1953

[6] Herrmann, S., Roynette, B.: *Boundedness and convergence of some self-attracting diffusions*, Math. Ann. 325(1): 81-96 (2003)

[7] Herrmann, S., Scheutzow, M., *Rate of convergence of some self-attracting diffusions*, Stochastic Processes and their Applications 111 (1): 41-55 (2004).

[8] Hsu, Elton P., *Stochastic Analysis on Manifold*, American Mathematical Society, Providence, RI, 2002.

[9] Ikeda, N., Watanabe, S.: *Stochastic Differential Equations and Diffusion Processes*, North Holland, New-York, 1981

[10] Karatzas, I., Shreve, S.E.: *Brownian motion and stochastic calculus*, Springer-Verlag, New-York, second edition, 1991

[11] Raimond, O.: *Self-attracting diffusions : Case of the constant interaction*, Probab. Theor. Relat. Fields 107, Issue 2: 177-196 (1997)

[12] Raimond, O.: *Self-interacting diffusions: a simulated annealing version*, Probab. Theory Relat. Fields 144 (1-2): 247-279 (2009)